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Resonance in nonlinear bubble oscillations

By MICHAEL S. LONGUET-HIGGINS

Center for Studies of Nonlinear Dynamics, La Jolla Institute, PO Box 1434, La Jolla, CA 92038, USA *and* Institute for Nonlinear Science, University of California, San Diego, R-002, La Jolla, CA 92093, USA

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In two recent papers (Longuet-Higgins 1989*a*, *b*) the author showed that the shape oscillations of bubbles can emit sound like a monopole source, at second order in the distortion parameter ϵ. In the second paper (LH2) it was predicted that the emission would be amplified when the second harmonic frequency $2\sigma_n$ of the shape oscillation approaches the frequency ω of the breathing mode. This 'resonance' would however be drastically limited by damping due to acoustic radiation and thermal diffusion. The predictions were confirmed by further numerical calculations in Longuet-Higgins (1990*a*).

Ffowcs Williams & Guo (1991) have questioned the conclusions of LH2 on the grounds that near resonance there is a slow (secular) transfer of energy between the shape oscillation and the volumetric mode which tends to diminish the amplitude of the shape oscillation and hence falsify the perturbation analysis. They have also argued that the volumetric mode never grows sufficiently to produce sound of the stated order of magnitude. In the present paper we show that these assertions are unfounded. Ffowcs Williams & Guo considered only undamped oscillations. Here we show that when the appropriate damping is included in their analysis the secular transfer of energy becomes completely insignificant. The resulting pressure pulse (figure 5 below) is found to be essentially identical to that calculated in LH2, figure 3. Moreover, in the initial-value problem considered in LH2, the excitation of the volumetric mode takes place not by a secular energy transfer but by a resonance during the first few cycles of the shape oscillation. This accounts for the amplification near resonance found in Longuet-Higgins (1990*a*). Finally, it is pointed out that the initial energy of the shape oscillations is far greater than is required to produce the $O(\epsilon^2)$ volume pulsations that were studied in LH2, and which were used for a comparison with field data. This acoustic radiation was not calculated by Ffowcs Williams & Guo.

1. Introduction

The emission of underwater sound by newly formed bubbles has been a subject of intense interest since a conference held in Lerici, Italy in July 1987; see Kerman (1988). In two recent papers (Longuet-Higgins 1989*a*, *b*) hereinafter referred to as LH1 and LH2) the author pointed out that the shape oscillations of bubbles (which must be stimulated during bubble formation) will necessarily be accompanied by the emission of a monopole component of radiated sound. This component has twice the frequency σ_n of the shape oscillation, and its amplitude is initially proportional to ϵ^2, where ϵ is a parameter measuring the distortion of the bubble from its equilibrium, spherical shape.

So long as the shape oscillation is not in resonance with the radial, 'breathing mode' of the bubble, that is to say provided $(2\sigma_n/\omega)$ is away from unity, where ω is the breathing-mode frequency, there is no exchange of energy between the shape oscillation and the breathing mode, and the emitted sound remains of order ϵ^2. This is true whether or not all dissipation of energy due to thermal diffusion, acoustic radiation or viscous forces is neglected, or is included. In practice, for comparison with oceanic noise the inclusion of dissipation is essential; see LH2, §§4–8.

LH2 considered an initial-value problem in which the fluid was at rest and the initial form was the sum of pure shape oscillations, with zero difference in volume from the equilibrium value. Without dissipation, the amplitudes of the shape oscillations contain a factor $(4\sigma_n^2-\omega^2)$ in the denominator. This suggests an increase in the 'response' as $2\sigma_n/\omega \to 1$. Obviously the theory cannot be taken as far as $2\sigma_n/\omega = 1$. However, with the inclusion of dissipation (see LH2 §§4–8) the factor is replaced by

$$[(4\sigma_n^2-\omega^2)^2+(4\gamma_n\sigma_n)^2]^{\frac{1}{2}},$$

where γ_n is a non-zero damping parameter. This makes the denominator always positive, and renders the theory uniformly valid in regard to the frequency σ_n, provided that $\gamma_n/\sigma_n \gg \epsilon$.

Ffowcs Williams & Guo (1991) have questioned the conclusions of LH2 on the basis of a purely undamped analysis of bubble oscillations. They have directed their comments exclusively to the undamped analysis in §3 of LH2. (This was used by the present author only as an introduction to the subsequent theory of fully damped oscillations.) Ffowcs Williams & Guo (hereinafter referred to as FWG) point out that near to resonance there tends to be a slow, secular transfer of energy between the initial shape oscillations and the volumetric mode. Such a periodic energy exchange, and its description in terms of a two-timescale analysis is quite familiar in nonlinear wave theory, including the theory of surface waves (Benney 1962; Bretherton 1964). FWG claim that the secular transfer vitiates the conclusions of LH2, although in fact §3 was not used in any of the applications to real bubbles!

In the present paper we examine the question of the secular transfer of energy by introducing appropriate damping terms into the inviscid analysis of FWG. We find (see §8) that this has the effect of suppressing the long-term transfer of energy to the volumetric mode. In the example considered by FWG, that of a single distortion mode of degree $n = 6$, the secular transfer of energy to the volumetric mode is found to be completely negligible. The amplitude and shape of the resulting pulse of sound (see figure 5) are shown to be quite unlike those given by the undamped theory (figure 1*a*) but closely resemble the pulse calculated in LH2, figure 3(*a*), for a similar initial-value problem. This is a strong indication that the damped theory of LH2 is in fact correct.

How then should one account for the increase in acoustical energy near resonance, found in Longuet-Higgins (1990*a*; LH3) by numerical calculations? The evident explanation is that the excitation of the volumetric mode occurs during the first few cycles of the shape oscillation. A discussion of the analogous case of a simple harmonic oscillator subject to a highly damped excitation is given in the Appendix to this paper.

Ffowcs Williams & Guo have also raised doubts as to whether there was sufficient energy in the shape oscillations to generate significant volumetric oscillations. In §9 we consider the energy budget and show that even with damping included, there is in fact many times more energy in the shape oscillations that is required to produce the volumetric oscillations predicted in LH2.

To avoid confusion it is important to distinguish between the $O(\epsilon^2)$ volume pulsations due to the shape oscillations, which are the subject of LH2 and LH3, and the $O(\epsilon)$ volume pulsations due to secular energy transfer (FWG). In the present problem the $O(\epsilon^2)$ volume pulsations are, paradoxically, much the greater.

Whether the $O(\epsilon^2)$ volume pulsations due to the shape oscillations do in fact make a significant contribution to the oceanic acoustical background is a separate question, which must be decided by the relative effectiveness of other noise-generating mechanisms. Since the appropriate value of ϵ is at present unknown, the answer will ultimately depend on comparisons with field observations, as discussed for example in LH2 and LH3. This answer will not affect the validity of the fluid dynamical calculations in LH2 and LH3.

The plan of the present paper is as follows. In §§2–6 we first carry the undamped theory of FWG to order ϵ^2 at finite time and show that it is consistent with the off-resonant calculations of LH2. Near resonance the amplitude of the volumetric mode is shown actually to increase by a large factor. At resonance itself the factor is $\sqrt{8}/\epsilon$, that is 28 in the case $\epsilon = 0.1$ considered by FWG.

In §§7 and 8 we extend the previous analysis to include realistic values of the damping, and demonstrate its drastic effect on the shape and amplitude of the emitted pulse. In §9 the balance of energy is considered, and a further discussion follows in §10.

2. Shape oscillations: inviscid theory

We consider the shape oscillations of a gas bubble of equilibrium radius a in an unbounded liquid of density ρ and ambient pressure p_A which is nearly atmospheric. The equations of motion and boundary conditions have been derived correct to second order in the shape parameter ϵ by LH1, and have been used by FWG. We shall follow mainly the notation of the earlier paper, but as in FWG it will be convenient to choose units of length, mass and time so that $a = 1$, $p_A = 1$, $\rho = 1$. The inertia of the gas in the bubble is ignored.

The radial displacement η of the bubble surface and the velocity potential Φ for the motion in the water are expanded formally in the scheme

$$\left.\begin{aligned} \eta &= \epsilon\eta' + \epsilon^2\eta'' + \dots, \\ \Phi &= \epsilon\Phi' + \epsilon^2\Phi'' + \dots, \end{aligned}\right\} \tag{2.1}$$

where ϵ is a small perturbation parameter, and if t denotes the (normalized) time we write

$$\tau = \epsilon t \tag{2.2}$$

for the 'slow' time. r, θ and ϕ denote spherical coordinates with origin at the centre of the sphere.

The equations of motion and boundary condition for the linear approximation $\epsilon\eta'$ admit solutions in the form of normal modes:

$$\left.\begin{aligned} \eta'_n &= \tfrac{1}{2}(C_n\,\mathrm{e}^{\mathrm{i}\sigma_n t} + C_n^*\,\mathrm{e}^{-\mathrm{i}\sigma_n t})\,S_n, \\ \Phi'_n &= -\frac{\mathrm{i}\sigma_n}{2(n+1)}(C_n\,\mathrm{e}^{\mathrm{i}\sigma_n t} - C_n^*\,\mathrm{e}^{-\mathrm{i}\sigma_n t})\frac{S_n}{r^{n+1}}, \end{aligned}\right\} \tag{2.3}$$

where $S_n(\theta, \phi)$ is a spherical harmonic of degree n, $C_n(\tau)$ is a complex amplitude and the radian frequency σ_n is given by

$$\sigma_n^2 = \left\{\begin{aligned} &(n-1)(n+1)(n+2)\,T, && n > 0, \\ &3\gamma + (3\gamma - 1)\,2T, && n = 0, \end{aligned}\right\} \tag{2.4}$$

where T denotes surface tension and $1 < \gamma < 1.4$. We shall also write $\sigma_0 = \omega$.

We consider an initial perturbation in which, at time $t = 0$, the fluid is at rest and the volume perturbation is zero, to second order. Thus in the first approximation we assume

$$\left.\begin{aligned} \eta' &= \tfrac{1}{2}(C_0\,\mathrm{e}^{\mathrm{i}\omega t} + C_n\,\mathrm{e}^{\mathrm{i}\sigma_n t}) + \text{c.c.}, \\ \Phi' &= -\frac{\mathrm{i}\omega}{2r} C_0\,\mathrm{e}^{\mathrm{i}\omega t} - \frac{\mathrm{i}\sigma_n}{2(n+1)} C_n\,\mathrm{e}^{\mathrm{i}\sigma_n t} \frac{S_n}{r^{n+1}} + \text{c.c.}, \end{aligned}\right\} \tag{2.5}$$

where
$$C_0(0) = 0, \quad C_n(0) = 1 \tag{2.6}$$

and c.c. denotes the complex conjugate terms.

In the next approximation it can be found either from LH1 or from FWG that

$$\eta'' = \eta_0'' + \eta_n'', \tag{2.7}$$

where η_0'' satisfies

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega^2\right)\eta_0'' = \frac{1}{8(2n+1)}\left[\frac{4n+9}{n+1}\sigma_n^2 - 2\omega^2\right] C_n^2\,\mathrm{e}^{2\mathrm{i}\sigma_n t} - \mathrm{i}\omega\frac{\mathrm{d}C_0}{\mathrm{d}\tau}\mathrm{e}^{\mathrm{i}\omega t} + Q_0 + \text{c.c.}, \tag{2.8}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \sigma_n^2\right)\eta_n'' = \tfrac{1}{4}[3\sigma_n^2 - 3\sigma\omega - (n-1)\,\omega^2]\,C_0\,C_n^*\,\mathrm{e}^{\mathrm{i}(\omega-\sigma_n)t} - \mathrm{i}\sigma_n\frac{\mathrm{d}C_n}{\mathrm{d}\tau} + Q_n + \text{c.c.} \tag{2.9}$$

and where

$$\begin{aligned} 4Q_0 = {} & [\tfrac{3}{2}(\gamma+2)\,\omega^2 + (3\gamma-1)\,T]\,C_0^2\,\mathrm{e}^{2\mathrm{i}\omega t} + [\tfrac{3}{2}\gamma\omega^2 + (3\gamma-1)\,T]\,C_0\,C_0^* \\ & + \frac{1}{2(2n+1)}\left[\frac{3\omega_n^2}{n+1} - 2\omega^2\right] C_n\,C_n^*. \end{aligned} \tag{2.10}$$

A similar expression for $4Q_n$ will not be needed.

3. The monopole radiation

Our task now is to calculate the monopole pressure term at infinity (which is not done by FWG). Let us introduce the relative change in the bubble volume V from its equilibrium value V_0, that is

$$h = \frac{a}{3}\left(\frac{V}{V_0} - 1\right) \tag{3.1}$$

(see LH1, §5). In terms of h, the pressure fluctuation p_∞ is given by

$$p_\infty = \frac{a^2}{r}\frac{\partial^2 h}{\partial t^2} \tag{3.2}$$

(see LH2, §6). The initial conditions on h are that

$$h = 0, \quad \partial h/\partial t = 0 \quad \text{when} \quad t = 0. \tag{3.3}$$

Now, correct to second order in ϵ we have

$$h = \bar{\eta} + \frac{\overline{\eta^2}}{a} \tag{3.4}$$

where an overbar denotes the spherical average. So, setting $a = 1$,

$$h = \epsilon\overline{\eta'} + \epsilon^2(\overline{\eta''} + \overline{\eta'^2}). \tag{3.5}$$

The $O(\epsilon)$ initial conditions

$$\overline{\eta'} = 0, \qquad \frac{\partial \overline{\eta'}}{\partial t} = 0 \quad (t=0) \tag{3.6}$$

are already satisfied, and the $O(\epsilon^2)$ initial conditions are

$$\overline{\eta''} = -\overline{\eta'^2}, \qquad \frac{\partial \overline{\eta''}}{\partial t} = 0 \quad (t=0) \tag{3.7}$$

by (3.6). We have then to solve equation (2.8) for η_0'' subject to the initial conditions that

$$\eta_0'' = -\overline{\eta'^2}, \qquad \frac{\partial \overline{\eta_0''}}{\partial t} = 0 \quad (t=0). \tag{3.8}$$

4. Exact resonance: $2\sigma_n = \omega$

By equating to zero the first two terms on the right-hand side (2.8) and of (2.9) one obtains

$$\left.\begin{aligned} \frac{\mathrm{d}C_0}{\mathrm{d}\tau} &= \mathrm{i}\omega P C_n^2, \qquad P = \frac{1}{32}\frac{(4n-1)}{(2n+1)(n+1)}, \\ \frac{\mathrm{d}C_n}{\mathrm{d}t} &= \mathrm{i}\omega Q C_0 C_n^*, \quad Q = \tfrac{1}{8}(4n-1), \end{aligned}\right\} \tag{4.1}$$

which have the solution

$$C_0 = \mathrm{i}A_0, \qquad C_n = A_n, \tag{4.2}$$

where
$$A_0 = \left(\frac{P}{Q}\right)^{\frac{1}{2}} \tanh\,[(PQ)^{\frac{1}{2}}\omega\tau], \qquad A_n = \operatorname{sech}\,[(PQ)^{\frac{1}{2}}\omega\tau] \tag{4.3}$$

(FWG, §6). These satisfy the initial conditions $A_0(0) = 0$, $A_n(1) = 1$. From (2.8) we have then to solve

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\eta_0'' = Q_0 \tag{4.4}$$

subject to the initial conditions

$$\eta_0'' = -\frac{1}{2n+1}, \qquad \frac{\partial \eta''}{\partial t} = 0. \tag{4.5}$$

Hence we find

$$\begin{aligned} \eta_0'' = \frac{1}{4}\left[(\gamma+2) + \tfrac{2}{3}(3\gamma-1)\frac{T}{\omega^2}\right] A_0^2 \cos 2\omega t + \frac{1}{4}\left[3\gamma + 2(3\gamma-1)\frac{T}{\omega^2}\right] A_0^2 \\ - \frac{(8n+5)}{16(2n+1)(n+1)} A_n^2 + \left[\frac{(8n+5)}{16(2n+1)(n+1)} - \frac{1}{2n+1}\right] \cos\omega t + \text{c.c.} \end{aligned} \tag{4.6}$$

Carrying through the calculation of p_∞ indicated in §3 we find, correct to second order in ϵ,

$$p_\infty = \frac{4T}{r}(n-1)(n+1)(n+2)F(\tau), \tag{4.7}$$

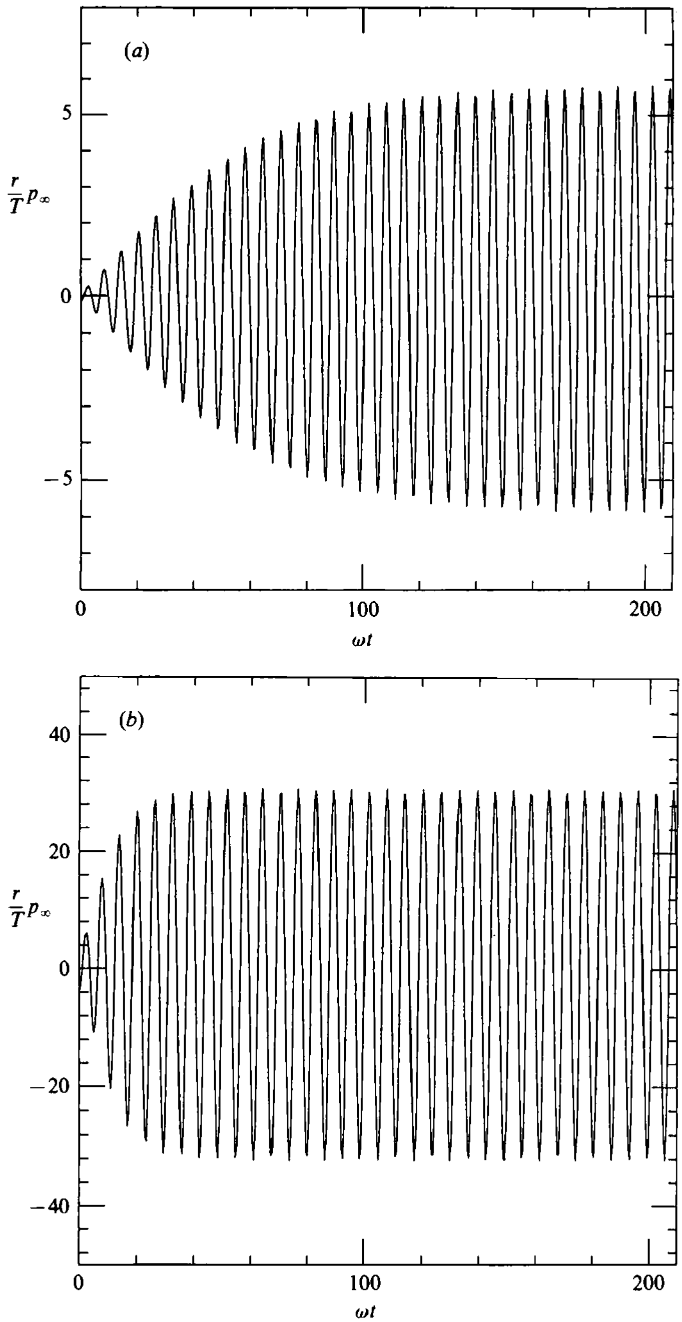

FIGURE 1. The shape of the emitted pressure pulse p' at distance r resulting from an initial distortion mode of degree $n = 6$. (a) $\epsilon = 0.1$, (b) $\epsilon = 0.5$.

where

$$F(\tau) = \epsilon A_0 \sin \omega\tau + \frac{\epsilon^2}{2n+1}\left[\frac{(8n+1)}{16(n+1)} - \frac{4n-1}{16(n+1)}A_n - \tfrac{1}{2}A_n^2\right]\cos\omega\tau$$

$$+\epsilon^2\left[\gamma + (2\gamma - \tfrac{1}{3})\frac{T}{\omega^2}\right]A_0^2 \cos 2\omega\tau. \tag{4.8}$$

As a check we note that when $t = 0$, the expression for the initial pressure is

$$(p_\infty)_{t=0} = -\frac{(n-1)^2(n+2)}{2n+1}\frac{\epsilon^2 T}{r}, \tag{4.9}$$

in agreement with LH2, equation (6.19). On the other hand as $\tau \to \infty$ the most important term in p_∞ is the term in ϵ, that is

$$(p_\infty)_{t=\infty} \sim 4(n-1)(n+1)(n+2)\frac{\epsilon T A_0}{r}\sin\omega t, \tag{4.10}$$

where from (4.3)

$$A_0 = \left(\frac{P}{Q}\right)^{\frac{1}{2}} = \frac{1}{2(n+1)^{\frac{1}{2}}(2n+1)^{\frac{1}{2}}}. \tag{4.11}$$

The proportional amplification is therefore

$$R = \frac{|p_\infty|_{t\to\infty}}{(p_\infty)_{t=0}} \sim \frac{2(n+1)^{\frac{1}{2}}(2n+1)^{\frac{1}{2}}}{\epsilon(n-1)}. \tag{4.12}$$

It will be seen that the ratio of the final amplitude of the breathing mode to the initial amplitude of the shape oscillation is, for large n,

$$\frac{A_0(\infty)}{A_n(0)} = \left(\frac{P}{Q}\right)^{\frac{1}{2}} \sim \frac{1}{\sqrt{8}}n^{-1} \tag{4.13}$$

by (4.11). For large n, this is only a small quantity, as was noted by FWG.

By contrast, the ratio of the corresponding pressure oscillations, from (4.12) is

$$R \sim \sqrt{8}\epsilon^{-1}, \tag{4.14}$$

which for small values of ϵ can be quite large.

Figure 1 (*a*) showed a plot of p_∞ as a function of ωt in the case $n = 6$ and $\epsilon = 0.1$ discussed by FWG. For clean water at room temperature the corresponding bubble radius is about 0.02 cm. It can be seen how the pressure oscillation grows to its limiting value in about 20 cycles.

Figure 1 (*b*) shows the corresponding pressure trace when $\epsilon = 0.5$. The initial and final values of $|p_\infty|$ are both greater than when $\epsilon = 0.1$, and the rate of amplification is about 5 times as fast. The final amplification, however, is not so large.

It will be noted that we have evaluated C_0 only to lowest order in ϵ. It would be possible to write, say,

$$\epsilon C_0 = \epsilon C_0' + \epsilon^2 C_0'' + \dots$$

and then the secular terms in C_0'' could be determined by means of third-order terms on the right-hand side of an equation analogous to (2.8). This would give an $O(\epsilon^2)$

correction to the long-term behaviour of ϵC_0. However, the initial conditions (4.5) for C_0' would not be affected. Hence (4.14), for example, remains valid to lowest order in ϵ. Further, for positive, $O(1)$ values of the time τ, since $C_0'''(0) = 0$, we see that $\epsilon^2 C_0'''(\tau)$ is of order ϵ^3. Hence terms involving C_0'' make only an $O(\epsilon^3)$ contribution to the pressure.

In other words, at all finite, positive times τ the above analysis is correct to order ϵ^2. For longer times it is correct to order ϵ, which is all that is required.

5. Near-resonance: $2\sigma_n/\omega = 1+O(\epsilon)$

In the more general case of near-resonance, when $(2\sigma_n/\omega-1)$ is of order ϵ, we expect a periodic exchange of energy between the two modes, the time-behaviour of the amplitudes C_0 and C_n being described by elliptic functions, as in Bretherton (1964), for example. For the bubble oscillation problem, FWG considered the special case

$$\frac{2\sigma_n}{\omega} = (1+\epsilon)^{-1} \sim 1-\epsilon \tag{5.1}$$

and obtained the maximum breathing-mode amplitude as

$$(A_0)_{\max} = \frac{2\epsilon}{4n-1}\left[\left(1+\frac{(4n-1)^2}{16(n+1)(2n+1)}\right)^{\frac{1}{2}}-1\right] \tag{5.2}$$

supposing that $(A_n)_{\max} = A_n(0) = 1$. The corresponding pressure oscillation, to order ϵ, will be of amplitude given by

$$r|p_\infty| = \epsilon(A_0)_{\max}\,\omega^2 = 4\epsilon T(n-1)(n+1)(n+2)(A_0)_{\max}. \tag{5.3}$$

Now the pressure p_∞ at time $t = 0$ is given by (4.9), independently of the breathing-mode frequency. Hence the relative pressure amplification R in this case is given by

$$R(\epsilon) = \frac{8(n+1)(2n+1)}{\epsilon(n-1)(4n-1)}\left[\left(1+\frac{(4n-1)^2}{16(n+1)(2n+1)}\right)^{\frac{1}{2}}-1\right] \tag{5.4}$$

to lowest order in ϵ. This is to be compared with the amplification $R(0)$ at resonance, which is given by (4.12).

In the more general case when

$$\frac{2\sigma_n}{\omega} = 1+\lambda\epsilon, \quad -1 \leqslant \lambda \leqslant 1 \tag{5.5}$$

we note that the relative amplification $R(\lambda\epsilon)$ is an analytic function of λ, with a maximum when $\lambda = 0$. Hence we have approximately

$$R(\lambda\epsilon) = R(0)+\lambda^2[R(\epsilon)-R(0)]. \tag{5.6}$$

This enables us to plot the relative amplification R as a function of $2\sigma_n/\omega$ over the range

$$1-\epsilon \leqslant \frac{2\sigma_n}{\omega} \leqslant 1+\epsilon, \tag{5.7}$$

as shown by the upper curves in figures 2(*a*, *b*).

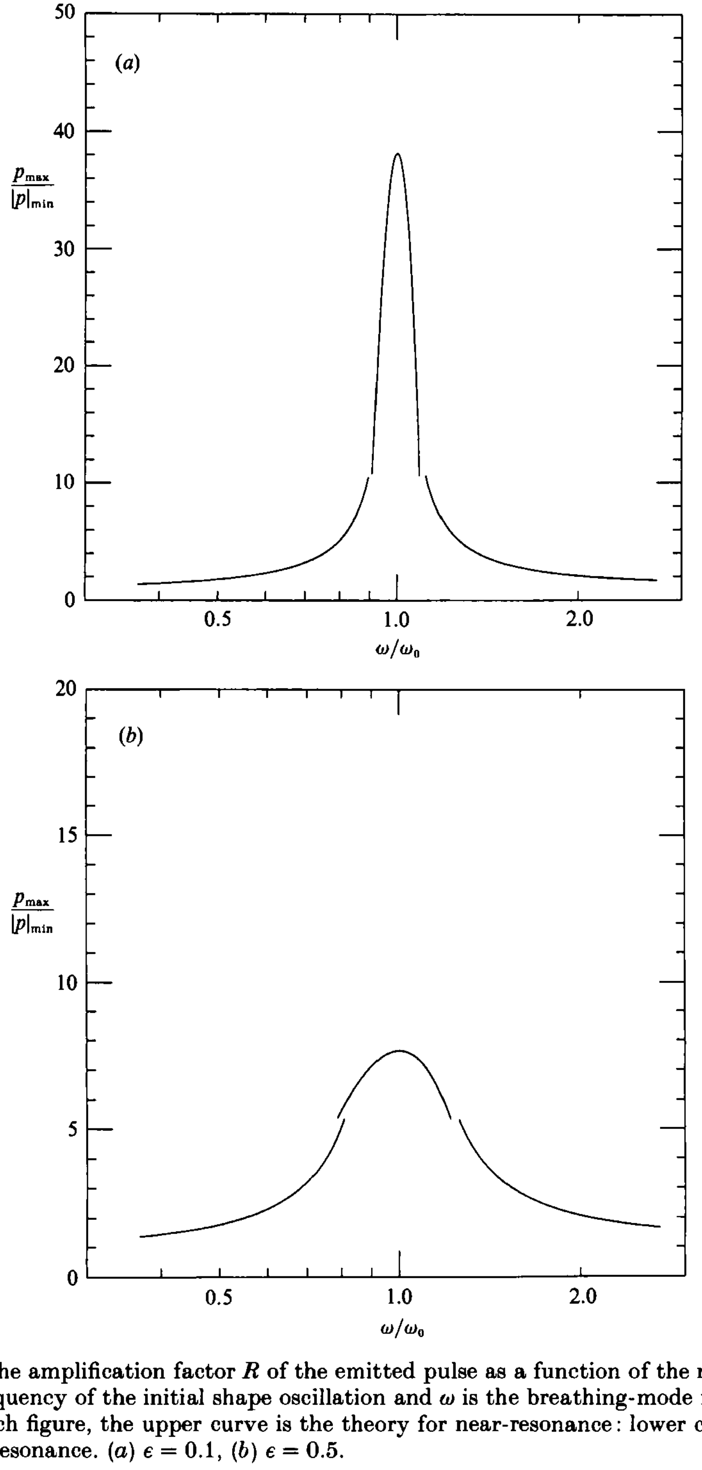

FIGURE 2. The amplification factor R of the emitted pulse as a function of the ratio $2\sigma_n/\omega$, where σ_n is the frequency of the initial shape oscillation and ω is the breathing-mode frequency, and for $n = 6$. In each figure, the upper curve is the theory for near-resonance: lower curve is the theory away from resonance. (a) $\epsilon = 0.1$, (b) $\epsilon = 0.5$.

6. Away from resonance: $2\sigma_n/\omega = 1+O(1)$

In the case when $(2\sigma_n/\omega-1)$ is of order ϵ^0, that is to say away from resonance, there is no transfer of energy between the modes and we may solve (2.8) and (2.9) for η_0'' and η_n'' directly, assuming C_0 and C_n to be constants. The result is identical to the solution given in LH2, §3, for the inviscid case. In fact from (3.20) and (3.21) we have, when $C_n = 1$, $C_0 = 0$,

$$p_\infty = \frac{1}{r}(P_n \cos 2\sigma_n t + Q_n \cos \omega t), \tag{6.1}$$

where

$$\left.\begin{aligned} P_n &= \frac{4\sigma_n^2}{4\sigma_n^2-\omega^2}\frac{(n-1)(n+2)(4n-1)}{4(2n+1)}T, \\ Q_n &= -\frac{\omega^2}{4\sigma_n^2}P_n + \frac{3(n-1)(n+2)}{4(2n+1)}T. \end{aligned}\right\} \tag{6.2}$$

The solution (6.1) then consists of two components, one of frequency $2\sigma_n$ resulting directly from the shape oscillation (see LH1, §7) and the other, of frequency ω, being a free radial oscillation stimulated at the initial instant.

When $t = 0$, the two components are in-phase, and we have

$$(p_\infty)_{t=0} = \frac{1}{r}(P_n + Q_n), \tag{6.3}$$

which is found to reduce directly to the expression (4.9). Note that the factor $(4\sigma_n^2-\omega^2)$ in the denominator of P_n and Q_n disappears when they are summed. On the other hand when the two components are in antiphase we have

$$|p_\infty| = \frac{1}{r}|P_n - Q_n|, \tag{6.4}$$

which is greater than (6.3) since it contains the factor $(4\sigma_n^2-\omega^2)$ in the denominator of each term. The amplification is expressed by the ratio

$$R = \left|\frac{P_n - Q_n}{P_n + Q_n}\right|. \tag{6.5}$$

This function is shown as the lower curves in figures 2 (*a*, *b*). Both figures correspond to $n = 6$. When $\epsilon = 0.1$ (figure 2*a*) the off-resonance curve intersects the near-resonance curve from §5 fairly smoothly. Although in the transition region neither theory is completely valid, nevertheless the combined curves convincingly demonstrate a resonant effect.

When $\epsilon = 0.5$ the two theories join less smoothly and the resonance is broader.

7. Effects of damping

So far we have ignored the loss of energy by damping, but from the analysis given in LH2, §§4–6, we may expect the damping to play a dominant role in the evolution of the acoustical pulse. Already from figure 3 (*a*) of LH2 we can see that for a bubble radius of $a = 0.02$ cm, corresponding nearly to $n = 6$, the pulse has the form of a damped oscillation with a decrement of 25% per cycle, as opposed to an initial growth of 30% per cycle predicted by the inviscid theory of figure 1(*a*) above.

The main sources of damping, as discussed in LH2, §§4 and 5, are viscosity, thermal diffusivity and acoustic radiation, the last two being dominant.

For a very rough analysis we return to (2.8) and (2.9) and replace $\mathrm{d}/\mathrm{d}t$ everywhere by $(\mathrm{d}/\mathrm{d}t+\gamma_i)$, where γ_i is an appropriate damping coefficient. Then $\mathrm{d}/\mathrm{d}\tau$ must be replaced by $(\mathrm{d}/\mathrm{d}\tau+\gamma_i')$, where $\gamma_i' = \gamma_i/\epsilon$. In the case of 'exact resonance', for example, (4.1) becomes

$$\left.\begin{aligned}\frac{\mathrm{d}C_0}{\mathrm{d}\tau} &= -\gamma_0' C_0 + \mathrm{i}\omega P C_n^2,\\ \frac{\mathrm{d}C_n}{\mathrm{d}\tau} &= -\gamma_n' C_n + \mathrm{i}\omega Q C_0 C_n^*.\end{aligned}\right\} \tag{7.1}$$

Setting
$$C_0 = \mathrm{i}A_0, \quad C_n = A_n \tag{7.2}$$
as before, we have
$$\left.\begin{aligned}\frac{\mathrm{d}A_0}{\mathrm{d}\tau} &= -\gamma_0' A_0 + \omega P A_n^2,\\ \frac{\mathrm{d}A_n}{\mathrm{d}\tau} &= -\gamma_n' A_n - \omega Q A_0 A_n.\end{aligned}\right\} \tag{7.3}$$

Given appropriate values of γ_0' and γ_n' we may solve (7.3) for $A_0(\tau)$ and $A_n(\tau)$ numerically, with the initial conditions $A_0(0) = 0$, $A_n(0) = 1$.

To determine appropriate values of γ_0' we note that $C_0(\tau)$ will be subject to at least the radiation, viscous and thermal losses discussed in LH2, §5. Thus

$$\gamma_0 = \gamma_{\mathrm{R}} + \gamma_{\mathrm{V}} + \gamma_{\mathrm{TH}} \tag{7.4}$$

where γ_{R}, γ_{V} and γ_{RH} are given by equations (5.4), (5.5) and (5.12) of LH2. Numerically, when $n = 6$ and $a = 0.01948$ we find

$$\gamma_0/\omega = 0.0316. \tag{7.5}$$

To determine γ_n we note that the shape oscillation is subject to both viscous losses from the linear mode and to viscous, thermal and radiation losses from the monopole, even when $C_0 = 0$. The first are of order ϵ^2, the second of order ϵ^4. Adopting the rough analysis of LH2, §6, we have from equation (6.6) of LH2,

$$2\gamma_n = -\frac{1}{A_n^2}\frac{\mathrm{d}A_n^2}{\mathrm{d}t} = \frac{\alpha_1 + \alpha_2(\epsilon A_n)^2}{\beta_1 + 2\beta_2(\epsilon A_n)^2} \tag{7.6}$$

where $\alpha_1, \alpha_2, \beta_1$ and β_2, are constant coefficients, with α_1, β_1 representing linear ($O(\epsilon)$) terms and $\alpha_2, 2\beta_2$ nonlinear ($O(\epsilon^2)$) terms in the dissipation and the total energy respectively. Thus γ_n is amplitude-dependent, in general, varying monotonically between $\alpha_1/2\beta_1$ for small amplitudes and $\alpha_2/4\beta_2$ for sufficiently large amplitudes. In the case $n = 6$, $\epsilon = 0.1$ we find that

$$0.809 < \gamma_n/\omega < 22.0, \tag{7.7}$$

the lower value representing the nonlinear damping. The main reason why γ_n is generally larger than γ_0 is because of the smaller initial energy of the shape oscillation.

Inserting these values into (7.3) and carrying out the numerical computation we find the curves shown in figures 3 and 4. It will be seen that A_n falls rapidly to zero,

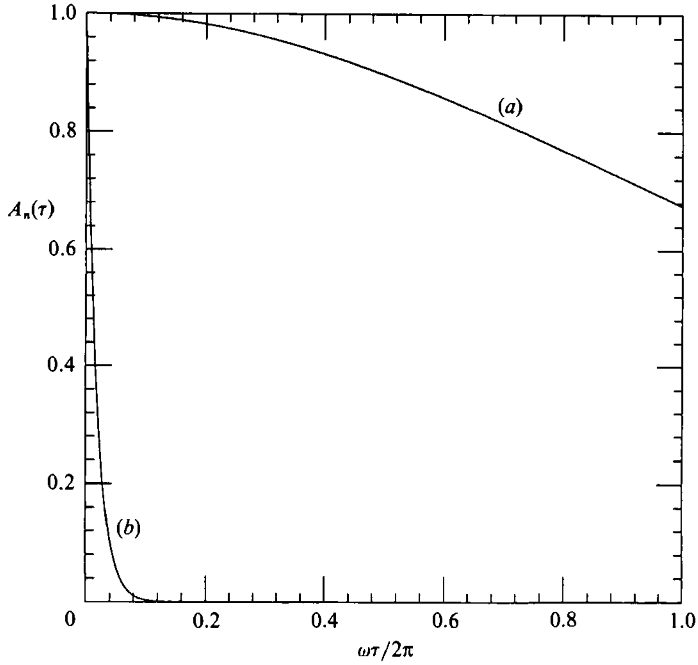

FIGURE 3. The behaviour of $A_n(\tau)$ as a function of $\omega\tau/2\pi$ when $n = 6$, and $\epsilon = 0.1$. (*a*) No damping (as in FWG, figure 3), (*b*) including damping.

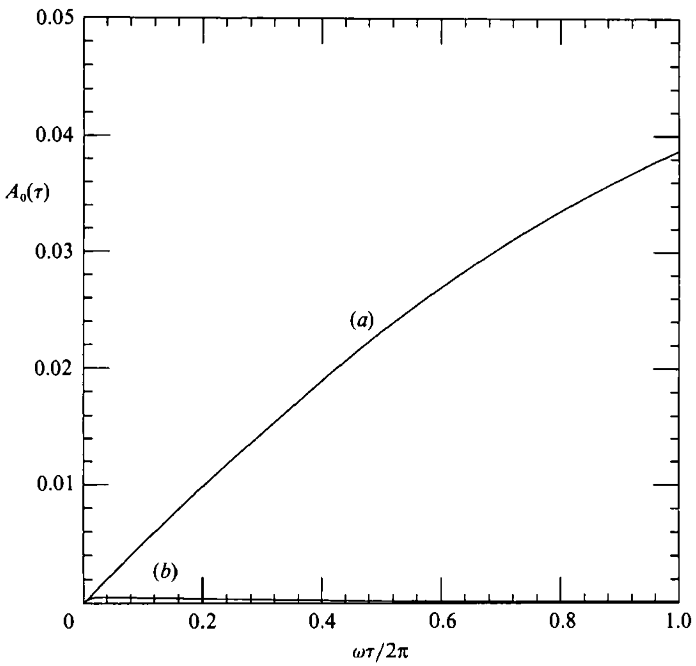

FIGURE 4. The behaviour of $A_0(\tau)$ as a function of $\omega\tau/2\pi$ when $n = 6$ and $\epsilon = 0.1$. (*a*) No damping (cf. FWG, figure 3), (*b*) including damping.

under the influence of the strong damping. A_0 grows at first but only to a maximum of 0.00051, less than 1% of its greatest value in the undamped case. It then falls to zero more slowly than A_n.

This behaviour contrasts strongly with the undamped case, shown in figure 3, which is equivalent to figure 3 of FWG. This suggested that all the energy in A_n was transferred to A_0. In practice, the energy is lost by damping before any appreciable transfer can take place.

8. The pressure pulse

To complete the calculation of the acoustical pressure we now solve the damped equation

$$\left(\frac{\partial^2}{\partial t^2}+2\gamma_0\frac{\partial}{\partial t}+\omega^{*2}\right)\eta_0'' = Q_0 \tag{8.1}$$

corresponding to (4.4) where ω^* is a modified frequency, subject to the initial conditions (4.5).

Carrying through the analysis correct to order ϵ, and remembering that γ/ω^* is formally of order ϵ, we obtain eventually for the pressure p_∞ as $r\to\infty$,

$$p_\infty = \frac{4T}{r}(n-1)(n+1)(n+2)F^*(t) \tag{8.2}$$

as in (4.6), except that now

$$F^*(t) = \epsilon A_0 \sin\omega t+\epsilon^2\left[\frac{8n+1}{16(n+1)}\mathrm{e}^{-\gamma_0 t}-\frac{4n-1}{16(n+1)}A_n-\tfrac{1}{2}A_n^2\right]\cos\omega^* t$$
$$+\epsilon^2\left[\gamma+\frac{(2\gamma-\frac{1}{3})T}{\omega^2}\right]A_0^2\cos 2\omega^* t. \tag{8.3}$$

Thus the only difference, formally, lies in the occurrence of the damping factor $\mathrm{e}^{-\gamma_0 t}$ multiplying the term in $\cos\omega^* t$ which is independent of A_n. This term arises from the initial conditions (4.5) at time $t=0$.

The calculated pressure pulse p_∞ is shown in figure 5, in the case $n = 6$, $\epsilon = 0.1$. The comparable case without damping is shown in figure 2(*a*). The contrast is evident. In the real (damped) case, the form of p_∞ is nearly a simple, exponentially damped sinewave corresponding to the term in $\mathrm{e}^{-\gamma_0 t}$ in (8.3). Thus

$$p_\infty \doteqdot \epsilon^2\frac{T}{4r}(n-1)(8n+1)(n+2)\mathrm{e}^{-\gamma_0 t}\cos\omega^* t. \tag{8.4}$$

Physically, this term arises mainly from the second-order ($O(\epsilon^2)$) 'squeeze' applied to the bubble by the initial shape distortion. The shape of the damped pulse is very similar to that calculated by LH2, figure 3(*a*). In that case there were several harmonics of comparable magnitude in the initial shape, but only the harmonic $n = 6$ was close to resonance. The transfer of energy to the breathing mode, and the exchange between the other shape oscillations was ignored.

Our handling of the damping by means of (8.2) is admittedly rough. We note however that the effects of damping on the forcing function Q_0 are already taken care of by the damping terms in (7.3). Any third-order terms that might, in undamped

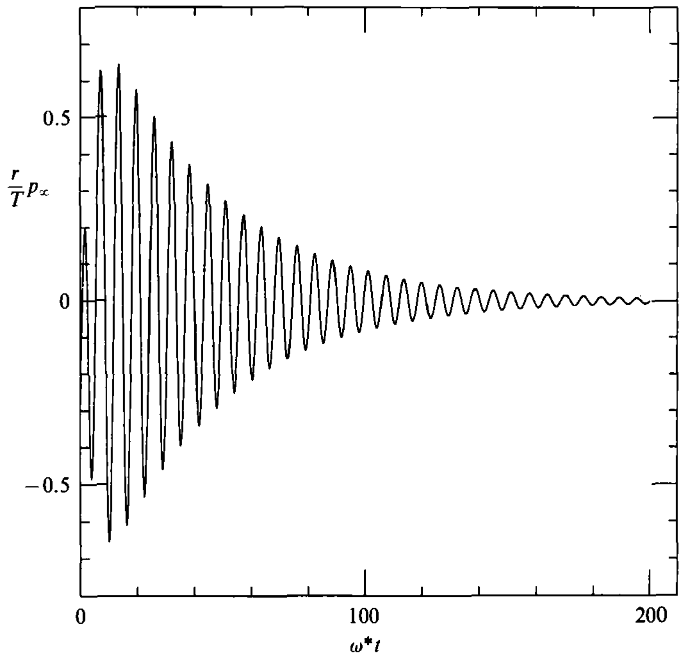

FIGURE 5. The calculated pressure pulse p_∞ due to an initial shape distortion, with damping: $n = 6$, $\epsilon = 0.1$. See figure 2(*a*) for the same initial conditions, but without damping.

circumstances, be present on the right-hand side of (8.3) are likely to be strongly damped also. In view of the rapid decay of both A_n and A_0, and the relatively small magnitude of the latter at all times, the third-order terms seem unlikely to be significant. Hence the long-term behaviour will still be given by (8.4), approximately.

9. The energy balance

Ffowcs Williams & Guo (1991) have claimed that the analysis of LH2 violates energy constraints, and that the energy of the shape oscillations is insufficient to maintain significant volume pulsations. We shall now examine this question.

The total energy in the initial shape oscillation $\eta = \epsilon S_n \cos \sigma_n t$, being twice the kinetic energy, is

$$E_n = 2\pi \frac{\epsilon^2 \sigma_n^2}{(n+1)(2n+1))} \tag{9.1}$$

to lowest order in ϵ. Similarly the total energy in the breathing mode $\eta = B \cos \omega t$ is

$$E_0 = 2\pi B^2 \omega^2. \tag{9.2}$$

Thus when $\omega = 2\sigma_n$ we have

$$\frac{E_n}{E_0} = \frac{1}{4(n+1)(2n+1)} \frac{\epsilon^2}{B^2}. \tag{9.3}$$

Now the amplitude B of the breathing mode is related to the pressure amplitude $|p_\infty|$ by

$$B = \frac{p_\infty}{r\omega^2} = \frac{p_\infty}{4r\sigma_n^2}. \tag{9.4}$$

The initial value of p_∞ is given by (4.9), regardless of the subsequent damping (see LH2, §6). Since σ_n^2 is given by (2.4) above, we have

$$B = \frac{(n-1)\,\epsilon^2}{4(n+1)\,(2n+1)} \tag{9.5}$$

and so from (9.3) we have initially

$$\frac{E_n}{E_0} = \frac{4(n+1)\,(2n+1)}{(n-1)^2\epsilon^2} \sim \frac{8}{\epsilon^2} \tag{9.6}$$

for $n \gg 1$. This ratio is very large, being of order 800 in the example $\epsilon - 0.1$ considered above.

Without damping, we have seen that the amplitude of the breathing mode would ultimately increase in the ratio (4.12), so that E_n/E_0 tends to 1 and all the energy is transferred to the breathing mode.

In the real, damped case we see from figure 5 that the maximum amplitude of the breathing mode is about twice its initial value, so that the ratio E_n/E_0 is still of order $2/\epsilon^2$, or 200. Thus we see that there is more than enough energy available for the volumetric oscillations. Nearly all of it is dissipated by damping, principally due to acoustical radiation and thermal conductivity. The effective energy balance is not between the two types of oscillation, as claimed by FWG, but between the oscillation energies and their rates of dissipation.

The shape and amplitude of the pulse in Figure 5 are altogether comparable to the pulse shown in figure 3(*a*) of LH2 in a similar case but where more than one mode was present. For the calculation of the real, damped pulses in figures 3 and 6 of LH2 the author used the rough energy balance equation (6.5) of LH2 which balances the decrease in energy for each mode against the corresponding rate of dissipation. From the above it is quite clear that energy constraints were not in fact violated, as was claimed by FWG. We note that FWG do not refer to those parts of LH2 (most of the paper) in which damping was taken into account, and which were used as a basis for comparison with ocean noise spectra.

10. Discussion and conclusions

In order to determine the effect of energy transfer between a shape oscillation and the fundamental 'breathing mode', we have considered two formal initial-value problems, one without damping and one with damping.

The problem without damping, studied by FWG, is somewhat academic. But the solution shows that although the maximum amplitude of the breathing mode is quite small, -0.03 times the amplitude of the initial shape oscillation, nevertheless the emitted pressure pulse is relatively *large* compared to the pulse at time $t = 0$. In fact the pulse is magnified by a factor of about $\sqrt{8}/\epsilon$, that is about 28.3, when $\epsilon = 0.1$. Hence it can be said that a spectral resonance does indeed exist. Moreover, when the amplification of the pulse is studied in relation to its initial value, as in figure 2, it is found that the near-resonant theory of FWG is contiguous with the away-from-resonance theory of LH2 in which there is formally no energy exchange.

The second, more realistic problem includes damping, as in LH2, as well as any resonant transfer of energy between the modes. The most striking feature is the very rapid decay of the shape oscillation (figure 3) and hence the negligible growth of the

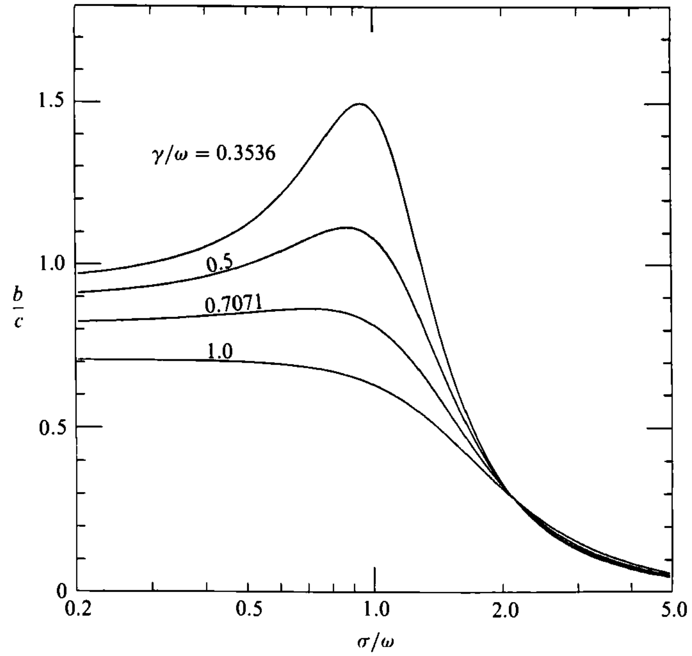

FIGURE 6. Relative amplitude of the long-term response of a lightly damped oscillator to a highly damped forcing function.

breathing mode (figure 4). The direct contribution to the pressure pulse from the breathing mode is quite insignificant. The dominant contribution to the pulse comes from the ringing of the bubble as a result of the initial squeeze from the shape distortion. This produces a tone decaying exponentially at a comparatively low rate.

It follows that in practice the shape of the pulse is dominated not by resonant energy transfer between the modes but on the contrary by the damping, which arises mainly from acoustic radiation and thermal diffusion.

The initial amplitude of the pulse is of order $\epsilon^2 T/r$, as was noted in LH2, §6. This is of the same order or larger than the amplitude used in an estimate of the acoustical energy flux in a typical oceanic situation (LH2, §8). The estimated acoustical energy is proportional to ϵ^4, and so depends strongly on the appropriate value of ϵ. However, for a given ϵ, the estimates of the acoustical flux are hardly at all affected by any energy exchange between the modes.

The question arises: is the degree of damping of the shape oscillation so great that it becomes meaningless to speak of a 'resonance' between it and the breathing mode? To answer this, we consider in the Appendix the response $z(t)$ of a simple linear oscillator, with natural frequency ω and damping γ_0, to a highly damped forcing function of $\exp[\mathrm{i}(\sigma+\mathrm{i}\gamma)t]$, $t>0$, where $\gamma \gg \gamma_0 > 0$. (When $t<0$, $z(t)$ is zero.) It is found that at large positive times t the response (t) tends to an exponentially damped sine-wave $b\,\mathrm{e}^{\mathrm{i}(\omega+\mathrm{i}\gamma_0 t}$, just as in (8.3). The ratio $(b\omega^2/a)$ is shown in figure 6 as a function of σ/ω. It will be seen that $b\omega^2/a$ has a maximum at around $\sigma/\omega = 1$, so long as $\gamma/\omega < 1$, approximately. This is when the time constant for the forcing function is comparable to the period of the response.

For applications to the ocean, where bubbles are created close to a pressure release surface, the radiation damping is reduced, owing to cancellation of the sound source by its 'image'. The effective damping is therefore less, by a factor of order 2. Hence resonant amplification will be more pronounced near the surface than in the interior of the fluid (bubble coalescence or splitting). However, the long-term transfer of energy between modes remains negligible.

A recent calculation of the sound emitted by the shape oscillations of a near-surface bubble has been carried out, using relevant values of the damping (Longuet-Higgins 1990*a*). For a bubble of given initial shape, the total energy of the pulse was indeed shown to be augmented for those bubble-sizes such that $2\sigma_n/\omega$ was close to 1.

In the light of our findings it is appropriate to review some of the stated conclusions of Ffowcs Williams & Guo (1991).

First, in the Abstract of their paper and also in §8, the authors speak of 'volume pulsations' as though they were associated solely with the radial, breathing mode of oscillation. We have seen, however, both in §3 onwards and also in LH1 and LH2, that volume pulsations are an essential accompaniment of shape oscillations at order ϵ^2. Moreover, these $O(\epsilon^2)$ volume oscillations are excited first near time $t = 0$, before any resonant exchange of energy with the breathing mode can take place. The statement by FWG that 'the volumetric pulsation has very small amplitude in comparison with that of the initial distortion' is thus seen not to be relevant to the main question, which is whether the $O(\epsilon^2)$ volume pulsations are capable of producing a significant contribution to the underwater sound spectrum.

A second theme of FWG is that, in the non-dissipative theory, the scope for amplification of the $O(\epsilon^2)$ volume pulsations near the critical frequencies when $2\sigma_n/\omega \approx 1$ is limited by a consideration of the total energy. In the inviscid case this is obvious, though there is indeed a tendency for the amplification of the pressure pulse to increase towards $2\sigma_n/\omega = 1$, even away from resonance (see figure $2a, b$, above) without violation of energy conservation.

We assume, however, that we are primarily interested in the emission of sound by real bubbles, in which case the loss of energy by dissipation and radiation is the dominating effect. Thus, for real bubbles, equations of energy conservation became irrelevant, except as a remote upper bound; it is the degree of damping that is crucial. The statement by FWG that 'a direct perturbation approach to this problem fails to conform with the principle of energy conservation' is thus completely misleading.

Moreover, it has to be emphasized that in the application of the theory to the real ocean that was considered in LH2, it was the fully damped theory that was used. The estimates were based only on the $O(\epsilon^2)$ terms, excited mainly at the initial instant of the pulses. These estimates, contained in §§5–8 of LH2, are now seen to be unaffected by the work of FWG. Remarkably, FWG do not mention these estimates, or the basis on which they were made, leaving the impression that they were based on an inviscid theory, and accordingly were too high.

Finally we note that in this paper we have examined only one particular type of initial condition: an initial distortion of the bubble, with zero initial velocity. In practice there will be other initial conditions also. For example, in the case of noise from bubbles entrained by raindrops, it has been shown (Longuet-Higgins 1990*b*) that the initial inward velocity of the fluid at the instant of bubble closure supplies most of the energy of oscillation. If mixed types of initial conditions apply, their effect will, to first order, be superposed linearly. Hence one would expect that in practice observed spectra of bubble noise will display peaks that are relatively less prominent than for the initial conditions discussed here.

It can also be foreseen that if the amplitude of the breathing mode is initially large, significant energy may be lost from the breathing mode to the shape oscillations, where it will be absorbed by the relatively high damping. In this way the breathing mode may suffer an anomalously high rate of damping, at least initially. This could lead to a dip in the noise spectrum at that particular frequency.

For the discussion of bubble noise spectra derived from oceanic field observations and laboratory data the reader is referred to Longuet-Higgins (1990*a*).

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Appendix. The response of a linear oscillator to a highly damped input

Consider the solution of the linear equation

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+2\gamma_0\frac{\mathrm{d}z}{\mathrm{d}t}+\omega^2 z = \begin{Bmatrix} 0, & t<0 \\ c\omega^2\,\mathrm{e}^{\mathrm{i}(\sigma+\mathrm{i}\gamma)t}, & t>0 \end{Bmatrix} \tag{A 1}$$

where $\gamma \gg \gamma_0 > 0$, with the initial conditions

$$z=0, \quad \frac{\mathrm{d}z}{\mathrm{d}t}=0 \quad \text{when} \quad t \leqslant 0. \tag{A 2}$$

The substitution $z=\mathrm{e}^{-\gamma_0 t}z'$ reduces (A 1) to the simpler problem

$$\frac{\mathrm{d}^2z'}{\mathrm{d}t^2}+\omega'^2 z' = \begin{Bmatrix} 0, & t<0 \\ c\omega^2\,\mathrm{e}^{\mathrm{i}(\sigma+\mathrm{i}\gamma')\tau}, & t>0 \end{Bmatrix} \tag{A 3}$$

where
$$\omega'^2=\omega^2-\gamma_0^2, \quad \gamma'=\gamma-\gamma_0 \tag{A 4}$$

and with the same initial conditions on z' as on z. The solution when $t>0$ is clearly

$$z'=\frac{c\omega^2}{\omega'^2-(\sigma+\mathrm{i}\gamma')^2}\left[\mathrm{e}^{(\mathrm{i}(\sigma+\mathrm{i}\gamma')\tau}-\cos\omega'\tau+(\gamma'-\mathrm{i}\sigma)\frac{\sin\omega' t}{\omega'}\right], \tag{A 5}$$

the real part of the right-hand side being taken. The first term in the square brackets, representing the forced oscillation, decays very rapidly, and we are left with the sum of the two sinusoidal terms, which combine to a sine-wave of amplitude

$$b=\frac{c\omega^2[P^2+(P\gamma'+Q\sigma)^2/\omega'^2]^{\frac{1}{2}}}{P^2+Q^2}, \tag{A 6}$$

where
$$P=\omega'^2+\gamma'^2-\sigma^2, \quad Q=2\gamma'\sigma. \tag{A 7}$$

Now consider the case when $\gamma_0/\omega \ll 1$ and γ/ω is $O(1)$ or greater. Then we may write $\omega'=\omega$ and $\gamma'=\gamma$ approximately, and b/c becomes a function of the two variables

$$\xi=(\sigma/\omega)^2 \quad \text{and} \quad G=(\gamma/\omega)^2 \tag{A 8}$$

alone, that is
$$\frac{b}{c}=\frac{[(1-G-\xi)^2+G(1+G+\xi)^2]^{\frac{1}{2}}}{(1+G-\xi)^2+4G\xi}. \tag{A 9}$$

Figure 6 shows b/c as a function of $\sigma/\omega=\xi^{\frac{1}{2}}$ for various values of $\gamma/\omega=G^{\frac{1}{2}}$. It will be seen that when $\gamma/\omega=1$, that is, the forcing function is 'critically damped', b/c has no positive maximum. However for $\gamma/\omega > 1$ there is a maximum when $\xi = 1-G$, as may be seen by writing (A 9) in the form

$$\frac{b}{c} = \left(\frac{1+G}{\eta^2+4G}\right)^{\frac{1}{2}}, \quad \eta = \xi - 1 + G. \tag{A 10}$$

The maximum value of b/c is thus

$$\left(\frac{b}{c}\right)_{\max} = \left(\frac{1+G}{4G}\right)^{\frac{1}{2}} = \frac{(\gamma^2+\omega^2)^{\frac{1}{2}}}{2\gamma}, \tag{A 11}$$

which tends to infinity as $\gamma/\omega \to 0$.

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